2000 Maritime Mathematics Competition
Solutions

1. At a meeting, one mathematician remarked to another, "There are nine fewer of us here than twice the product of the two digits of our total number." How many mathematicians were at the meeting?

   Lors d'une réunion de mathématiciens, un des participants remarque que le nombre total de personnes présentes à la réunion est neuf de moins que deux fois le produit des deux chiffres formant ce nombre. Combien de personnes ont assisté à la réunion?

   Solution:

   Let xy be the two digits of the number of mathematicians at the meeting. The mathematician's statement translates to

   $$10x + y + 9 = 2xy$$

   so

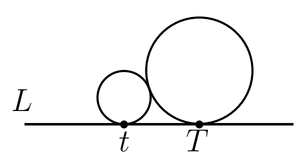
   $$y = \frac{10x + 9}{2x - 1}\,.$$

   When $x = 4$, we obtain $y = 49/7 = 7$. All other possibilities for x yield inadmissable values for y.

   There were, therefore, 47 mathematicians at the meeting.

2. Suppose that two circles with radii r and R intersect in a single point and that the straight line L is tangent to both circles at t and T respectively, as in the diagram below. Determine the distance between the points t and T.

   Si deux cercles de rayons r et R se coupent en un seul point, et la droite L est tangente aux deux cercles en t et T respectivement, tel qu'indiqué dans la figure ci-dessous, quelle est la distance entre les points t et T?



   Solution:

   Let o be the centre of the circle which contains the point t. Let O be the centre of the other circle. Then $oOTt$ is a quadrilateral with right angles at t and T. Moreover, we have

   $$|ot| = r, \quad |OT| = R, \quad \text{and} \quad |oO| = r + R.$$

Let point A on OT be such that $|AT| = r$. Now $|tT| = |oA|$, and by the Pythagorean Theorem,

$$|oA|^2 = |oO|^2 - |AO|^2 = (r + R)^2 - (R - r)^2 = 4rR.$$

Therefore, the distance between the points t and T is

$$\sqrt{4rR} = 2\sqrt{rR}\ .$$

3. There are 120 four digit numbers that contain only the digits 1, 2, 3, 4, 5, each at most once. Find the sum of all such numbers.

   Trouver la somme de tous les nombres à quatre chiffres dont les chiffres sont choisis, sans répetition, parmi 1, 2, 3, 4, 5. (Il y en a 120.)

   Solution:

   Let S be the set consisting of the 120 four digit numbers that may be formed using the digits 1, 2, 3, 4, 5, each digit being used at most once. By symmetry, each digit appears as the units digits in an element of S the same number of times, namely $120/5 = 24$. Therefore, the sum of the units digits taken over all elements of S is

$$24 \times (1 + 2 + 3 + 4 + 5) = 360.$$

   Similarly, the sum of all the tens digits is 360, as are the sums of the hundreds and thousands digits. Therefore, the sum of all the numbers in S is

$$360 + 360(10) + 360(100) + 360(1000) = 360 \times 1111 = 399,960\ .$$

4. A cubic box with edges 1 metre long is placed against a vertical wall. A ladder $\sqrt{15}$ metres long is placed so that it touches the wall as well as the free horizontal edge of the box. Find at what height the ladder touches the wall.

   Une boîte cubique d'un mètre d'arête est placée contre un mur vertical. Une échelle longue de $\sqrt{15}$ mètres est appuyée contre le mur de telle sorte qu'elle s'appuie également contre l'arête libre du cube. À quelle hauteur l'échelle touche-t-elle au mur?

   Solution:

   Let x be the height at which the ladder touches the wall. Let y be the distance between the foot of the ladder and the wall. By the

Pythagorean Theorem, $x^2 + y^2 = (\sqrt{15})^2 = 15$, and, using similar triangles,

$$\frac{x-1}{1} = \frac{1}{y-1} \quad \Longrightarrow \quad (x-1)(y-1) = 1 \quad \Longrightarrow \quad xy = x + y\ .$$

Now

$$15 = x^2 + y^2 = (x^2 + 2xy + y^2) - 2xy = (x+y)^2 - 2(x+y)$$

so, letting $z = x + y$, we have

$$z^2 - 2z - 15 = 0 \quad \Longrightarrow \quad (z-5)(z+3) = 0 \quad \Longrightarrow \quad z = 5 \quad \text{or} \quad z = -3\ .$$

As x and y are both positive, $z = -3$ is inadmissable. Therefore, $5 = z = x + y$ so $y = 5 - x$. Substituting into $xy = x + y$, we obtain

$$x(5-x) = 5 \quad \Longrightarrow \quad x^2 - 5x + 5 = 0.$$

By the Quadratic Formula,

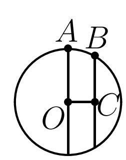
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(5)}}{2(1)} = \frac{5 \pm \sqrt{5}}{2}$$

so there are two solutions. The ladder touches the wall at a height of either $(5 + \sqrt{5})/2$ metres or $(5 - \sqrt{5})/2$ metres.

5. A circular grass plot 12 metres in diameter is cut by a straight gravel path 3 metres wide, one edge of which passes through the centre of the plot. Determine the number of square metres in the remaining grass area.

   Une pelouse circulaire de 12 mètres de diamètre est traversée d'une allée de gravier de 3 mètres de large dont un des bords passe par le centre de la pelouse. Trouver l'aire du reste de la pelouse.

   Solution:



In the above diagram, O is the centre of the circle, A and B lie on the circle, and C is such that OC is the width of the path. Since OB is a radius of the circle, $|OB| = 6$. Moreover, $|OC| = 3$ so $\angle COB = \frac{\pi}{3}$ radians and $|BC| = 3\sqrt{3}$. Thus

$$\text{area of } \triangle COB = \frac{1}{2}(3)(3\sqrt{3}) = \frac{9\sqrt{3}}{2}\ .$$

Furthermore, $\angle AOB$ measures $\frac{\pi}{6}$ radians so

$$\text{area of sector } AOB = \pi(6)^2 \times \frac{\pi/6}{2\pi} = 3\pi.$$

Therefore

$$\text{area of path } = 2\left(3\pi + \frac{9\sqrt{3}}{2}\right) = 6\pi + 9\sqrt{3}$$

so the remaining grass area contains

$$\pi(6)^2 - (6\pi + 9\sqrt{3}) = 30\pi - 9\sqrt{3} \text{ square metres.}$$

6. Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following two conditions.

   - Each rectangle has the same number of white squares and black squares.
   - No two rectangles have the same number of squares.

   Find the maximum value of p for which such a decomposition is possible. For this maximum value of p, determine all corresponding decompositions of the chessboard into p rectangles.

   Considérons les décompositions d'un échiquier 8×8 en p rectangles, sans chevauchement, et telles que les conditions suivantes soient satisfaites.

   - *Chaque rectangle comporte le même nombre de cases blanches et de cases noires.*
   - *Il n'y a pas deux rectangles qui ont le même nombre de cases.*

   Trouver la valeur maximale de p pour laquelle une telle décomposition soit possible. Pour cette valeur maximale de p, déterminer toutes les décompositions correspondantes.

Solution:

Consider a decomposition of the chessboard into p non-overlapping rectangles subject to the two given conditions. Let $a_1, a_2, \ldots, a_p$ be the number of squares in the rectangles in the decomposition.

Because of the second condition, the a_i's are all distinct so we may assume, without loss of generality, that $a_1 < a_2 < \cdots < a_p$. Furthermore, each a_i is even, by the first condition.

We claim that $p \leq 7$. To show this, suppose to the contrary that $p \geq 8$. Then the number of squares covered by the rectangles in the decomposition is

$$a_1 + a_2 + \cdots + a_p \geq 2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 72$$

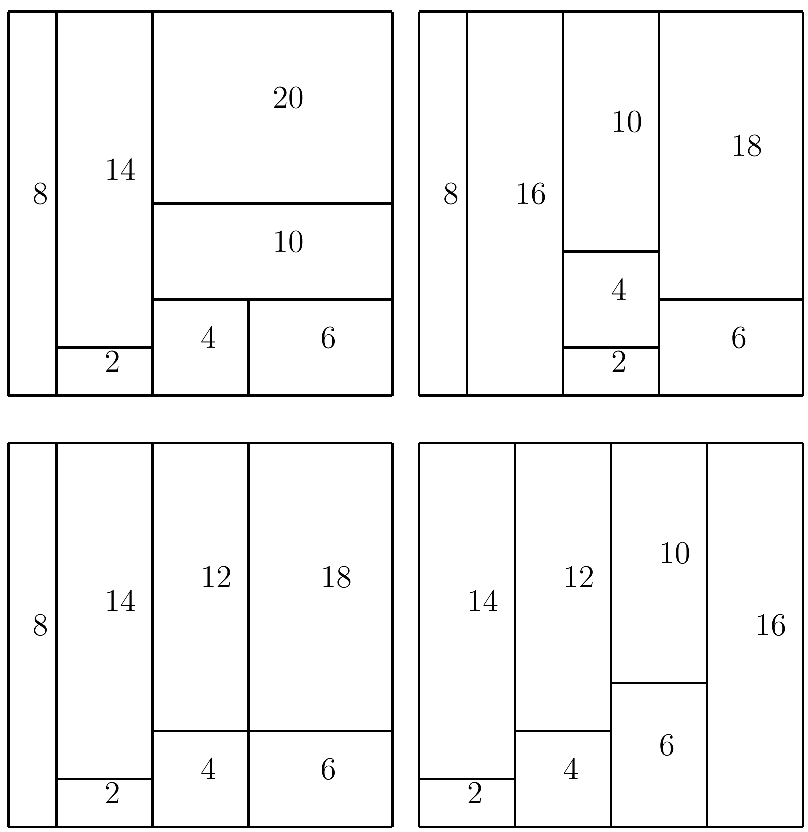
which is impossible since the chessboard has only 64 squares.

For $p = 7$, we obtain the following five sequences as the only possibilities for $(a_1, a_2, \ldots, a_7)$.

$$(2,4,6,8,10,12,22), (2,4,6,8,10,14,20), (2,4,6,8,10,16,18),$$

$$(2,4,6,8,12,14,18), (2,4,6,10,12,14,16)$$

To establish that the maximum value of p is indeed 7, it remains to show that there is an actual decomposition of the chessboard into 7 rectangles. Now no rectangle may have 22 squares since it is impossible to find a rectangle contained in the chessboard having dimensions 1×22 or 2×11. There is, therefore, no decomposition of the board corresponding to the first sequence. Each of the other 4 sequences, however, does have a corresponding decomposition as the diagram below illustrates.



Therefore, the maximum value of p is 7 and there are 4 decompositions of the chessboard into 7 non-overlapping rectangles subject to the given conditions.