1997 Maritime Mathematics Competition
Solutions

1 How many 1 cent coins (pennies), 5 cent coins (nickels), 10 cent coins (dimes) and 25 cent coins (quarters) would be worth 93 cents if there were eight coins altogether? Is there more than one solution?
Si on dispose exactement de 8 pièces de monnais, trouver les nombres de pièces d'un sou (pennies), pièces de 5 sous (nickels), pièces de 10 sous (dimes) et pièces de 25 sous (quarters) nécessaires pour former 93 cents. Y a-t-il plus qu'une seule solution ?

Solution: Let p, n, d, q be the numbers of pennies, nickels, dimes and quarters respectively. Then we are looking for positive integer solutions to the system

$$\begin{aligned} p + 5n + 10d + 25q &= 93 \\ p + n + d + q &= 8. \end{aligned}$$

Since the total is 93 cents, we must have 3 pennies, so $p = 3$. Substitution gives

$$\begin{aligned} 5n + 10d + 25q &= 90 \\ n + d + q &= 5 \end{aligned}$$

or

$$\begin{aligned} n + 2d + 5q &= 18 \\ n + d + q &= 5. \end{aligned}$$

Subtracting gives $d + 4q = 13$. The possibilities for the ordered pair (d, q) are therefore $(1, 3), (5, 2)$, and $(9, 1)$. The last two of these give totals over 90 cents, so the only possibility is $d = 1, q = 3, p = 3, n = 1$, that is, 3 quarters, 1 dime, 1 nickel and 3 pennies.

2 Find the value of xyz given that:
Trouver la valeur de xyz étant donnée :

$$\begin{aligned} x + y + z &= 1 \\ x^2 + y^2 + z^2 &= 2 \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

Solution: Squaring the first equation;

$$\begin{aligned} (x + y + z)^2 &= x^2 + y^2 + z^2 + 2(xy + yz + zx) \\ \text{so } xy + yz + zx &= -\frac{1}{2}. \end{aligned}$$

Cubing the first equation;

$$\begin{aligned}(x+y+z)^3 &= x^3+y^3+z^3+3(x+y+z)(xy+yz+zx)-3xyz\\ \text{so } 1 &= 3+3(1)\left(-\frac{1}{2}\right)-3xyz.\end{aligned}$$

Hence

$$xyz = \frac{1}{6}.$$

3 Show that for all natural numbers n:

Montrer que pour tout entier naturel n :

$$n! \leq \left(\frac{n+1}{2}\right)^n.$$

Note that/*À noter que:* $n! = (1)(2)(3)\cdots(n-1)(n)$

Solution: We will make use of the geometric-arithmetic mean inequality, that is,

$$ab \leq \left(\frac{a+b}{2}\right)^2.$$

Case 1: <u>n is even</u>: Pair off the numbers $1, 2, \ldots, n$ as $(1, n), (2, n-1), (3, n-2), \ldots, (k, n-k+1), \ldots (\frac{n}{2}, \frac{n}{2}+1)$. Using the above inequality on each pair,

$$\begin{aligned}1(n) &\leq \left(\frac{n+1}{2}\right)^2\\ 2(n-1) &\leq \left(\frac{n+1}{2}\right)^2\\ &\vdots \qquad \vdots\\ \left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) &\leq \left(\frac{n+1}{2}\right)^2\end{aligned}$$

and multiplying, since there are $\frac{n}{2}$ pairs:

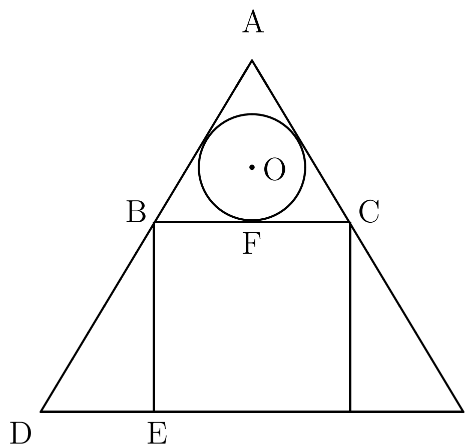
$$n! \leq \left(\frac{n+1}{2}\right)^n.$$

Case 2: <u>n is odd</u>: Proceed as above, but consider the $\frac{n-1}{2}$ pairs $(1, n), (2, n-1), (3, n-2), \ldots, (k, n-k+1), \ldots (\frac{n-1}{2}, \frac{n+1}{2}+1)$ and the middle number $\frac{n+1}{2}$. Using $\frac{n-1}{2}$ inequalities similar to those in Case 1,

$$\begin{aligned}\frac{n!}{\frac{n+1}{2}} &\leq \left(\frac{n+1}{2}\right)^{n-1}\\ \text{so } n! &\leq \left(\frac{n+1}{2}\right)^n.\end{aligned}$$

4 A circle and a square are inscribed inside an equilateral triangle as shown. If the sides of the triangle have unit length, what is the radius of the circle?

Un cercle et un carré sont inscrits à l'intérieur d'un triangle équilatéral, comme indiqué sur la figure. Quel serait le rayon du cercle si on suppose que la longueur des côtés du triangle est l'unité ?



Solution: Label the diagram as shown (O is the centre of the circle). Let x be the length of a side of the square, and let r be the radius of the circle.

Now $\triangle ABC$ is equilateral so $|AB| = |BC| = x$. Since AD has unit length, we have $|BD| = 1 - x$.

Consider now $\triangle BDE$. Since $\angle BDE = 60°$, we have

$$\frac{\sqrt{3}}{2} = \frac{|BE|}{|BD|} = \frac{x}{1-x}$$

implying that $x = \sqrt{3}/(2+\sqrt{3})$.

Now consider $\triangle OBF$. We observe that $\angle OFB$ is a right angle and that $\angle OBF = 30°$. Furthermore, we see that $|BF| = x/2$ by noting that the figure is symmetric about a vertical line through A. Thus

$$\frac{1}{\sqrt{3}} = \frac{|OF|}{|BF|} = \frac{r}{x/2}$$

so

$$r = \frac{x}{2\sqrt{3}} = \frac{1}{2(2+\sqrt{3})} = 1 - \frac{\sqrt{3}}{2}.$$

5 Let $p(x) = 1 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $a_1, a_2, \ldots, a_n$ are integers such that $a_1 + a_2 + \cdots + a_n$ is an even number. Show that there are no integer solutions to the equation $p(x) = 0$.
Soit $p(x) = 1 + a_1x + a_2x^2 + \ldots + a_nx^n$, où $a_1, a_2, \ldots, a_n$ sont des entiers tels que, $a_1 + a_2 + \ldots + a_n$ est pair. Montrer qu'ils n'existent pas de solutions entières (x entier) pour l'équation $p(x) = 0$.

Solution: If $p(x) = 0$ then $1 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$ so

$$x(a_1 + a_2x + \cdots + a_nx^{n-1}) = -1$$

but this implies that x divides -1 and the only integers that can do that are -1 and 1. If $x = 1$, then $p(x) = 1 + a_1 + a_2 + \cdots + a_n$ which is odd and so cannot be 0. If $x = -1$ then

$$\begin{aligned} p(x) &= 1 - a_1 + a_2 - \cdots + (-1)^n a_n \\ &= 1 + (a_1 + a_2 + \cdots + a_n) - 2(a_1 + a_3 + a_5 + \cdots) \end{aligned}$$

which is also odd and so cannot be 0. Thus there are no integer solutions to $p(x) = 0$.

6 A (circular) coin of radius r is dropped onto a floor which is tiled with equilateral triangles with sides of length ℓ. What is the probability that the coin comes to rest lying completely atop one tile with no overlap onto another tile?
On fait tomber une pièce de monnais circulaire de rayon r *sur un sol recouvert de triangles équilatérals avec des côtés de longueur ℓ. Quelle est la probabilité que la pièce se retrouve complétement à l'intérieur d'un triangle?*

Solution: There are two keys to this question. The first is to realize that the location of the center of the circle determines whether or not the coin lies completely inside one tile. Thus we need only consider the one triangle which contains the center of the circle. The circle will lie completely inside the tile if and only if the shortest distance from the center of the circle to the edge of the triangle is greater than r.

Let T_ℓ denote the equilateral triangle with side of length ℓ. Let $T_\ell(r)$ denote all the points inside the triangle whose distance to the edge of the triangle is greater than r. It is easy to convince yourself that $T_\ell(r)$ is also an equilateral triangle.

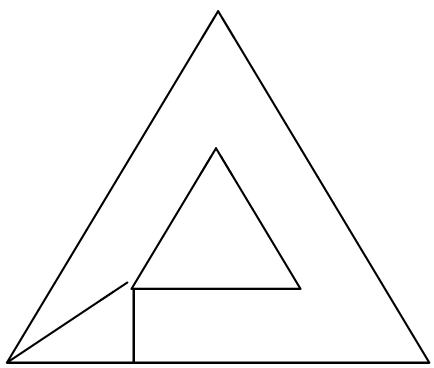
The second key is to realize that the probability of the coin lying completely inside one tile is equal to the probability that the center of the coin lies inside $T_\ell(r)$ and that this probability is equal to the proportion of T_ℓ which is filled up by $T_\ell(r)$, that is,

$$\frac{\text{area}(T_\ell(r))}{\text{area}(T_\ell)}.$$

Thus the problem is reduced to computing the areas of two equilateral triangles. Since the area of an equilateral triangle with sides of length ℓ is $\frac{\sqrt{3}l^2}{4}$ (slice the triangle down the middle, rearrange the two pieces into a rectangle and use Pythagorean Theorem) we have now reduced the problem to determining the length of a side of $T_\ell(r)$.

Consider the picture below of one tile: focus on the small triangle in the lower left corner of the tile. It has height r and if we let x denote the length of a side of $T_\ell(r)$ then it has base $\frac{\ell-x}{2}$. But the lower left angle of this triangle has a degree measure of 30° (by trigomometry or similar triangles) so:

$$\frac{\ell - x}{2r} = \sqrt{3}.$$



Solving for x we obtain that

$$x = \ell - 2\sqrt{3}r.$$

So, if $\ell \leq 2\sqrt{3}r$ there is no triangle and the probability is 0, otherwise the probability is

$$\frac{\left(\ell - 2\sqrt{3}r\right)^2}{l^2} = \left(1 - \frac{2\sqrt{3}r}{\ell}\right)^2.$$