1999 Maritime Mathematics Competition
Solutions

1. Let natural numbers be assigned to the letters of the alphabet as follows: A=1, B=2, C=3, ... , Z=26. The value of a word is defined to be the product of the numbers assigned to the letters in that word. For example, the value of MATH is $13 \times 1 \times 20 \times 8 = 2080$. Find a word whose value is 285.

   Faisons correspondre à chaque lettre de l'alphabet une valeur numérique comme suit: A=1, B=2, C=3, ... , Z=26. La valeur d'un mot sera le produit de la valeur de ses lettres. Par exemple, la valeur du mot MATH est $13 \times 1 \times 20 \times 8 = 2080$*. Trouver un mot dont la valeur est 285.*

   Solution: The number 285 factors as $3 \cdot 5 \cdot 19$ so the word must be made from letter 3 (C) , letter 5 (E) and letter 19 (S) and as many letter 1's (A) as you want, or letter 15 (O) and letter 19 (S) and as many letter 1's (A) as you want. Some possibilities in English are: CASE, ACES, SEC and SO.

2. A rhombus is a parallelogram with all four sides having the same length. If one of the interior angles of a rhombus is $60°$, find the ratio of the area of the rhombus to the area of the inscribed circle.

   Un losange est un parallélogramme dont les côtés sont égaux. Si un des angles internes d'un losange mesure $60°$*, trouver le rapport de l'aire du losange à l'aire du cercle inscrit.*

   Solution: Since we are just looking for the ratio of the two areas, we can assume that all sides of the rhombus have length 2. By dropping a line from a top corner of the rhombus at right angles to the base of the rhombus, we create a a $30° - 60° - 90°$ triangle. The ratios of the lengths of the sides of such a triangle are easily seen to be $2 : 1 : \sqrt{3}$. (Just cut an equilateral triangle down the middle.) so the height of the rhombus is $\sqrt{3}$. Thus the area of the rhombus is (base)(height) $= 2\sqrt{3}$. Now the key fact to note is that the diameter of the circle is the height of the rhombus (see diagram.) and the radius is half the diameter so the area of the circle is $\pi\left(\frac{\sqrt{3}}{2}\right)^2 = \pi\left(\frac{3}{4}\right)$. Hence the ratio of the area of the rhombus to the area of the inscribed circle is $2\sqrt{3} : \pi\left(\frac{3}{4}\right)$ or $1 : \frac{\sqrt{3}\pi}{8}$.

3. A straight line cuts the asymptotes of a hyperbola in points A and B and cuts the curve at points P and Q. Prove that $AP = BQ$. (Hint: Use the fact that every hyperbola can be rotated, translated and scaled so that it is given by the equation $xy = 1$, and the asymptotes in this case are just the x-axis and the y-axis.)

   Une droite coupe les asymptotes d'une hyperbole aux points A et B et coupe l'hyperbole en P et Q. Montrer que $AP = BQ$. (Indice: chaque hyperbole peut être convertit, par rotation, translation et contraction/dilation, en l'hyperbole d'équation $xy = 1$, et dans ce cas les asymptotes sont tout simplement l'axe des x et l'axe des y.)

   Solution: Let P have coordinates $(p, \frac{1}{p})$ and Q have coordinates $(q, \frac{1}{q})$. With no loss of generality let A be the point on the y-axis and B be the point on the x-axis which are also on the line through P and Q. The slope of the line through PQ is

$$\frac{\frac{1}{p} - \frac{1}{q}}{p - q} = -\frac{1}{pq}$$

   so the equation of the line is

$$y - \frac{1}{q} = -\frac{1}{pq}(x - q).$$

   Substituting $x = 0$ into this equation we obtain that $y = \frac{1}{p} + \frac{1}{q}$ so A has coordinates $(0, \frac{1}{p} + \frac{1}{q})$. Similarly, substituting $y = 0$ into the equation gives that $x = p + q$ so that B has coordinates $(p + q, 0)$. Now just compute:

$$\begin{aligned} AP &= \sqrt{\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{p}\right)^2 + (0 - p)^2} = \sqrt{\frac{1}{q^2} + p^2} \\ BQ &= \sqrt{\left(0 - \frac{1}{q}\right)^2 + (p + q - q)^2} = \sqrt{\frac{1}{q^2} + p^2}. \end{aligned}$$

4. Find the largest number n with the property that the sum of the cubes of its digits (in base 10) is greater than n.

   Trouver le plus grand nombre n dont la somme des cubes des chiffres (en base 10) est plus grand que n.

   Solution: If n has k digits then n is greater than or equal to 10^{k-1} and the sum of the cubes of its digits is less than or equal to 9^3k so we have that the number of digits must satisfy the inequality

   $$10^{k-1} \leq 9^3 k$$

   The largest value of k which satisfies this is $k = 4$ so n can have at most 4 digits so $n = abcd$ where a, b, c, d are numbers between 0 and 9. Next determine the largest that a can be: Refining the above inequality we have that

   $$a \cdot 1000 \leq n \leq a^3 + b^3 + c^3 + d^3 \leq a^3 + 3 \cdot 9^3 = a^3 + 2187$$

   This implies that a is 1 or 2. But if $a = 2$ then b must satisfy

   $$2000 + b \cdot 100 \leq n \leq a^3 + b^3 + c^3 + d^3 \leq 8 + b^3 + 2 \cdot 9^3 = b^3 + 1466$$

   or equivalently $b(b^2 - 100) \geq 534$,which is impossible for a digit between 0 and 9. Hence $a = 1$ and it can easily be checked that $n = 1999$ has the property so by the above, it must be the largest number with this property.

5. Find all nonnegative numbers x, y and z such that

   $$\begin{aligned} z^x &= y^{2x} \\ 2^z &= 2 \cdot 4^x \\ x + y + z &= 16. \end{aligned}$$

   Trouver tous les nombres non négatifs x, y et z satisfaisant aux équations ci-dessous.

   $$\begin{aligned} z^x &= y^{2x} \\ 2^z &= 2 \cdot 4^x \\ x + y + z &= 16 \end{aligned}$$

   Solution: The first equation can be rewritten as

   $$z^x = \left(y^2\right)^x$$

and since x is nonnegative we must have that either $x = 0$ or $z = y^2$.The second equation can be rewritten as

$$2^z = 2^{1+2x}$$

so $z = 1 + 2x$. If $x = 0$,then $z = 1$ and the last equation then tells us that $y = 15$. So $x = 0, y = 15, z = 1$ is one solution. If $x \neq 0$ then $x = \frac{1}{2}(z-1) = \frac{1}{2}(y^2-1)$.Substituting these into the last equation we have

$$\frac{1}{2}(y^2 - 1) + y + y^2 = 16$$

which simplifies to the quadratic

$$3y^2 + 2y - 33 = 0$$

which factors as

$$3y^2 + 2y - 33 = (3y + 11)(y - 3).$$

Since y is non-negative, we must have $y = 3$, so $z = 9$ and $x = 4$. So $x = 4, y = 3, z = 9$ is the only other solution.

6. The following symmetric table is known as Sundaram's sieve. The first row and column is the arithmetic progression $4, 7, 10, 13, \ldots$. Successive rows are also arithmetic progressions, the common differences, respectively, being the odd integers $3, 5, 7, 9, \ldots$. Show that for every positive integer n, $2n + 1$ is prime if and only if n is not in the table.

   La matrice symmétrique ci-dessous s'appelle le crible de Sundaram. La première rangée et la première colonne contiennent la suite arithmétique $4, 7, 10, 13, \ldots$. *Les rangées et colonnes successives du crible contiennent des suites arithmétiques dont les raisons sont les nombres impairs* $3, 5, 7, 9, \ldots$. *Montrer que pour tout nombre naturel* n, *le nombre* $2n+1$ *est premier si et seulement si* n *n'apparaît pas dans le crible.*

4	7	10	13	16	19	22	$\cdots$
7	12	17	22	27	32	37	$\cdots$
10	17	24	31	38	45	52	$\cdots$
13	22	31	40	49	58	67	$\cdots$
16	27	38	49	60	71	82	$\cdots$
19	32	45	58	71	84	97	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Solution: If an arithmetic progression starts at a and increases by a number d at each step, then the k^{th} number in the progression will be

$a+(k-1)d$. Using this fact one can show that, if we let a_{ij} denote the number in the i^{th} row and j^{th} column of the table, then

$$a_{ij} = i + j + 2ij.$$

To see this note that the first row is an arithmetic progression starting at 4 and increasing by 3 at each step, so the j^{th} number in the first row is given by

$$a_{1j} = 4 + (j-1)3 = 1 + 3j$$

Hence a_{ij} which is the i^{th} number in the arithmetic progression which is the j^{th} column, which starts at $a_{1j} = 1 + 3j$ and increases by $2j+1$ is given by

$$a_{ij} = 1 + 3j + (i-1)(2j+1) = i + j + 2ij.$$

Using this formula we can now prove the above.

Clearly n is in the table if and only if $n = a_{ij}$ for some numbers i and j which means that

$$2n + 1 = 2a_{ij} + 1 = 2i + 2j + 4ij + 1 = (2i+1)(2j+1)$$

so this is true if and only if $2n+1$ is a product of some pair of odd numbers and this is true if and only if $2n+1$ is not prime.