1999 Maritime Mathematics Competition

1. Let natural numbers be assigned to the letters of the alphabet as follows: $A = 1, B = 2, C = 3, \ldots, Z = 26$. The value of a word is defined to be the product of the numbers assigned to the letters in that word. For example, the value of $MATH$ is $13 \times 1 \times 20 \times 8 = 2080$. Find a word whose value is 285.

2. A rhombus is a parallelogram with all four sides having the same length. If one of the interior angles of a rhombus is $60°$, find the ratio of the area of the rhombus to the area of the inscribed circle.

3. A straight line cuts the asymptotes of a hyperbola in points A and B and cuts the curve at points P and Q. Prove that $AP = BQ$. (Hint: Use the fact that every hyperbola can be rotated, translated and scaled so that it is given by the equation $xy = 1$, and the asymptotes in this case are just the x-axis and the y-axis.)

4. Find the largest number n with the property that the sum of the cubes of its digits (in base 10) is greater than n.

5. Find all nonnegative numbers x, y and z such that

$$\begin{aligned} z^x &= y^{2x} \\ 2^z &= 2 \cdot 4^x \\ x + y + z &= 16. \end{aligned}$$

6. The following symmetric table is known as Sundaram's sieve. The first row and column is the arithmetic progression $4, 7, 10, 13, \ldots$. Successive rows are also arithmetic progressions, the common differences, respectively, being the odd integers $3, 5, 7, 9, \ldots$. Show that for every positive integer n, $2n + 1$ is prime if and only if n is not in the table.

$$\begin{array}{rrrrrrrl}
4 & 7 & 10 & 13 & 16 & 19 & 22 & \cdots \\
7 & 12 & 17 & 22 & 27 & 32 & 37 & \cdots \\
10 & 17 & 24 & 31 & 38 & 45 & 52 & \cdots \\
13 & 22 & 31 & 40 & 49 & 58 & 67 & \cdots \\
16 & 27 & 38 & 49 & 60 & 71 & 82 & \cdots \\
19 & 32 & 45 & 58 & 71 & 84 & 97 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$