2002 Maritime Mathematics Competition (Solutions)
Concours de Mathématiques des Maritimes 2002 (solutions)

1. "We'd better go! It's ten o'clock," exclaimed Rachel, looking at her watch. Jonathan looked at his watch. "Your watch must have stopped some time ago," he replied. "According to my watch, in 5 minutes, it will be as many minutes short of eleven as it was past ten o'clock 17 minutes ago."

   What was the time according to Jonathan's watch?

   « On devrait y aller! Il est dix heures », s'écria Rachel. Jonathan regarda sa montre. « Ta montre s'est arrêtée, je crois. », répondit-il. « Selon la mienne, il y a dix-sept minutes il était dix heures et un certain nombre de minutes. Dans cinq minutes, il sera onze heures moins le même nombre de minutes. »

   Quelle heure était-il selon la montre de Jonathan?

   Solution:

   Suppose that, according to Jonathan's watch, the time is x minutes past ten o'clock. Then there are $60 - x$ minutes until eleven o'clock so Jonathan's remark implies that

   $$(60 - x) - 5 = x - 17.$$

   Thus $2x = 72$ so $x = 36$. Therefore, according to Jonathan's watch, the time is 10:36.

2. A certain brand of hockey sticks priced at \$50 each was not selling well. When the store manager reduced the price per stick by a whole number of dollars, the whole remaining stock was sold for \$2002. What is the least number of items that could have been in stock?

   Les bâtons de hockey d'une certaine marque se vendent mal à 50 \$ la pièce. Le gérant du magasin réduit le prix par un nombre entier de dollars et par la suite le stock entier se vend pour 2002 \$. Quel est le plus petit nombre de bâtons que le stock pouvait contenir?

   Solution:

   Let x be the reduced price of a hockey stick, and let y be the number of sticks remaining in stock. Now x and y are positive integers with $x < 50$ and $xy = 2002$. Expressing 2002 as the product of its prime factors, we obtain $2002 = 2 \times 7 \times 11 \times 13$. Since $xy = 2002$, the smallest value of y is attained when x is as large as possible. Subject to the constraint that $x < 50$, we see that the maximum value for x is $x = 2 \times 13 = 26$. The corresponding value of y is $7 \times 11 = 77$ so the least number of sticks that could have been in stock is 77.

3. The points $(-6,1), (6,10), (9,6)$, and $(-3,-3)$ are the vertices of a rectangle. What is the area of the portion of this rectangle that lies above the x axis?

   Les points $(-6,1), (6,10), (9,6)$, et $(-3,-3)$ sont les sommets d'un rectangle. Quelle est l'aire de la partie de ce rectangle qui se trouve au dessus de l'axe des x?

   Solution:

   Label the vertices of the square as follows: $A(-6,1)$, $B(-3,-3)$, $C(9,6)$, and $D(6,10)$. Let E and F be, respectively, the points of intersection of the sides AB and BC with the x axis.

   The equation of the line through A and B is

$$\frac{y-1}{x-(-6)} = \frac{1-(-3)}{-6-(-3)} \quad \Longrightarrow \quad y = -\frac{4}{3}x - 7.$$

   The x intercept of this line is found by setting

$$-\frac{4}{3}x - 7 = 0 \quad \text{so} \quad x = -\frac{21}{4}\,.$$

   Therefore, E has coordinates $(-\frac{21}{4}, 0)$. Similarly, the equation of the line through B and C is

$$\frac{y-6}{x-9} = \frac{6-(-3)}{9-(-3)} \quad \text{so} \quad y = \frac{3}{4}x - \frac{3}{4}.$$

   The x intercept of this line is 1 so F has coordinates $(1,0)$.

   The length of EF is then $1 + \frac{21}{4} = \frac{25}{4}$ so

$$\text{area of } \triangle BEF = \frac{1}{2}\left(\frac{25}{4}\right)(3) = \frac{75}{8} = 9\frac{3}{8} \text{ square units.}$$

   Furthermore,

$$\text{length of } AB = \sqrt{(-6-(-3))^2 + (1-(-3))^2} = \sqrt{9+16} = \sqrt{25} = 5$$

   and

$$\text{length of } BC = \sqrt{(9-(-3))^2 + (6-(-3))^2} = \sqrt{144+81} = \sqrt{225} = 15$$

   so the area of the rectangle is $5 \times 15 = 75$ square units. Therefore, the area of the portion of the rectangle that lies above the x axis is $75 - 9\frac{3}{8} = 65\frac{5}{8}$ square units.

4. Find the smallest positive integer n such that

$$\sqrt{n} - \sqrt{n-1} < 0.01 \ .$$

Trouver le plus petit entier positif n tel que

$$\sqrt{n} - \sqrt{n-1} < 0.01 \ .$$

Solution:

The left hand side may be written as

$$\sqrt{n} - \sqrt{n-1} \times \frac{\sqrt{n}+\sqrt{n-1}}{\sqrt{n}+\sqrt{n-1}} = \frac{n-(n-1)}{\sqrt{n}+\sqrt{n-1}} = \frac{1}{\sqrt{n}+\sqrt{n-1}} \ .$$

The given inequality then becomes $\frac{1}{\sqrt{n}+\sqrt{n-1}} < 0.01$ which, by taking the reciprocal of each side, is equivalent to

$$\sqrt{n} + \sqrt{n-1} > 100.$$

Now $\sqrt{2500} = 50$ so

$$\sqrt{2500} + \sqrt{2499} < 100 \quad \text{and } \sqrt{2501} + \sqrt{2500} > 100.$$

Therefore, the smallest positive integer that satisfies the given inequality is $n = 2501$.

5. Consider the following number triangle. Each number is the sum of three numbers of the previous row; the number immediately above it and the numbers immediately to the right and left of that one. If no number appears in one or more of these locations, pretend that the vacant locations contain zeros. Prove that every row, beginning with the third row, contains at least one even number.

Dans le triangle ci-dessous, chaque nombre, sauf celui dans la première rangée, est la somme de trois nombres de la rangée précédente: celui qui ce trouve immédiatement au dessus de lui, et ceux qui se trouvent immédiatement à la droite et à la gauche de celui-ci. (On fait semblant que c'est un zero qui se trouve dans les positions vides.) Montrer que chaque rangée à partir de la troisième contient au moins un nombre pair.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 1 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Solution:

Notice that the first four entries in any row depend only upon the first four entries in the previous row. Now recording only the parity of each entry using 1 for odd and 0 for even, and starting with the third row, we obtain the following pattern.

$$\begin{array}{cccccccc} & & & & 1 & 0 & 1 & 0 \\ & & & 1 & 1 & 0 & 1 & \\ & & 1 & 0 & 0 & 0 & & \\ & 1 & 1 & 1 & 0 & & & \\ 1 & 0 & 1 & 0 & & & & \end{array}$$

The parity sequence of the seventh row is identical to that of the third row so the above pattern repeats. Since each row in the pattern contains a zero, there is an even integer in each row of the triangle.

6. Find all positive integers m, s, and t such that

$$m! = 4\,s! + 10\,t!\ .$$

(Note that $1! = 1, 2! = (1)(2) = 2, 3! = (1)(2)(3) = 6$, etc.)

Trouver toutes les solutions en entiers positifs m, s, et t de l'équation suivante:

$$m! = 4\,s! + 10\,t!\ .$$

(À noter que: $1! = 1, 2! = (1)(2) = 2, 3! = (1)(2)(3) = 6$, *etc.)*

Solution:

We note first of all that any solution (m, s, t) of the equation $m! = 4\,s! + 10\,t!$ must have $m > s$ and $m > t$. We now consider the following three cases: (i) $s = t$, (ii) $s > t$, and (iii) $s < t$.

(i) When $s = t$, the equation becomes $m! = 14s!$. Dividing by $s!$, we obtain

$$\frac{m!}{s!} = 14.$$

Notice that $\frac{m!}{s!}$ is either equal to m or equal to a product of consecutive positive integers. By virtue of the fact that 14 may not be expressed as a product of consecutive integers, we must have $m = 14$ and $s = 13$. Therefore, $m = 14, s = 13, t = 13$ is the unique solution in which $s = t$.

(ii) When $s > t$, dividing the equation by $t!$ gives

$$\frac{s!}{t!}\,[\frac{m!}{s!} - 4] = 10\ .$$

Let $x = \frac{s!}{t!}$. Since t is a positive integer, we must have $x \geq 2$. Moreover, as x is a factor of 10, the only possibilities are $x = 2, 5,$ or 10.

For $x = 2$, we must have $s = 2, t = 1$ and, $\frac{m!}{2!} - 4 = 5 \iff m! = 18$, which has no solution.

For $x = 10$, we must have $s = 10$, $t = 9$ and, $\frac{m!}{10!} - 4 = 1 \iff m! = (5)(10!)$, which has no solution.

For $x = 5$, we must have $s = 5$, $t = 4$ and, $\frac{m!}{5!} - 4 = 2 \iff m! = (6)(5!) \iff m = 6$. Therefore, $m = 6, s = 5, t = 4$ is the unique solution in which $s > t$.

(iii) Finally, when $s < t$, dividing the equation by $s!$ gives

$$\frac{t!}{s!}\,[\frac{m!}{t!} - 10] = 4\ .$$

Let $y = \frac{t!}{s!}$. Notice that we must have $y = 2$ or $y = 4$.

If $y = 2$, then $t = 2$, $s = 1$ and, $\frac{m}{2!} - 10 = 2$, which gives $m = 4$, and the solution $m = 4, s = 1, t = 2$ to the equation.

If $y = 4$, then $t = 4$, $s = 3$ and, $\frac{m!}{4!} - 10 = 1 \iff m! = (11)(4!)$, which is impossible.

To summarize, the given equation has three solutions, namely

$$(m, s, t) = (14, 13, 13), (6, 5, 4), (4, 1, 2)\ .$$