DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 2513

FINAL EXAMINATION Time: 3 HOURS
APRIL 1998

NO CALCULATORS. ALL QUESTIONS ARE OF EQUAL VALUE.

1. Given that $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, calculate the following:
   (a) $(2\mathbf{a} + \mathbf{b}) \cdot \mathbf{b}$,
   (b) the value of k such that $\mathbf{a} + k\mathbf{b}$ and $\mathbf{b}$ are orthogonal,
   (c) $\mathbf{a} \times \mathbf{b}$,
   (d) equation of the plane parallel to $\mathbf{a}$ and $\mathbf{b}$ and passing through the point $(1, 2, -1)$,
   (e) parametric equations of the line parallel to $\mathbf{a}$ and passing through the point $(1, 2, -1)$.

2. (a) Find an equation of the plane through the three points $A(1, -2, 3)$, $B(3, 1, 2)$ and $C(2, 3, -1)$.
   (b) Find a set of equations for the line which passes through the point $(1, -2, 3)$ and is parallel to the line of intersection of the two planes $x + y + z = 1$ and $3x - 2y + z = -4$.

3. (a) Suppose that the equation
$$x^3 + 3y^2 + z^2 - xy + 6y^2 z = 2$$
   defines z implicitly as a function of x and y, find $\dfrac{\partial z}{\partial x}$.
   (b) Using the chain rule, find $\dfrac{\partial z}{\partial r}$ if
$$z = \sqrt{x^3 + y^2}, \quad x = s^2 \tan r, \quad y = s^3 \sec r.$$
   (c) Show that $u(x, y) = \ln(x^2 + y^2)$ satisfies
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

4. (a) Find all local maxima, local minima and saddle points of
$$f(x, y) = x^3 + y^2 + 2xy + 4x + 4y.$$
   (b) Use Lagrange multipliers to find the minimum value of
$$f(x, y, z) = 3x^2 + 2y^2 + 3z^2$$
   subject to the constraint
$$12x + 4y - 6z + 17 = 0.$$

5. (a) Find the directional derivative of the function

$$F(x, y, z) = x^3 + xy^2 + z^2 + xyz.$$

at the point $P(1, 2, 1)$ in the direction of $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. In what direction does F change most rapidly at P?

(b) Let

$$f(x, y) = \sqrt{x^2 + y^3 + 1}.$$

i. Find an equation of the tangent plane to the surface $z = f(x, y)$ at the point $(4, 2, 5)$.

ii. Use differentials to approximate $f(4.01, 1.98)$.

6. (a) Use polar coordinates to evaluate

$$\iint_R \sqrt{x^2 + y^2}\, dA$$

where R is the region inside the circle $x^2 + y^2 = 2y$.

(b) Sketch the region of integration, reverse the order of integration and evaluate the integral

$$\int_0^1 \int_y^1 e^{x^2}\, dx\, dy.$$

7. (a) Use spherical coordinates to evaluate

$$\iiint_R x\, dV$$

where R is the region in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the coordinate planes.

(b) Use cylindrical coordinates to find the volume of the solid that the cylinder $x^2 + y^2 = 1$ cuts out of the sphere of radius 2 centred at the origin.

8. (a) Evaluate

$$\int_C z^2 y\, dx + xz\, dy + y\, dz$$

where C is the straight line segment from the origin to $(-1, 2, 3)$.

(b) Show that

$$\mathbf{F}(x, y) = (3x^2 + 4xy - 2y^2)\mathbf{i} + (2x^2 - 4xy - 3y^2)\mathbf{j}$$

is conservative and find a potential function ϕ, ie a function such that $\mathbf{F} = \nabla\phi$. Use ϕ to evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is any path from $(0, 0)$ to $(2, 1)$.