

DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 1003

FINAL EXAMINATION
DECEMBER 2000

TIME: 3 HOURS
TOTAL POINTS = 100

INSTRUCTIONS:

- (a) You must show all calculations for full marks.
- (b) Calculators **are not** permitted. (Huge calculations often mean you are on the wrong track!)
- (c) Note, however, that there is a choice in question 7. GOOD LUCK!

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MARKS

1. Find the derivatives of each of the following functions. (**Do not simplify answers.**)

(3) (a) $f(x) = \left(2x - \frac{1}{\sqrt{x}}\right)^3$

(3) (b) $f(x) = \frac{\arctan(x)}{x^2 - 2x + 3}$ (i.e. $\frac{\tan^{-1} x}{x^2 - 2x + 3}$)

(3) (c) $f(x) = e^{5x} \tan(4x)$

(3) (d) $f(x) = \ln(3x + \sin x)$

(3) (e) $f(x) = 2^{\cosh(3x)} + \frac{1}{\cos x}$

(3) (f) $f(x) = (x^2 + 1) \sin^{-1}(\sqrt{x})$ (i.e. $(x^2 + 1) \arcsin(\sqrt{x})$)

(3) (g) $g(x) = x^{3x}$

- (6) 2. (a) Use logarithmic differentiation to find $\frac{dy}{dx}$, where

$$y = \frac{(2x^2 - 1)^3 e^{5x}}{\sqrt{x^3 - 1}}.$$

- (6) (b) Find both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1$, $y = 2$ (in other notation, both y' and y'') if

$$y^2 + y = x^4 + 5.$$

- (10) 3. (a) Use the limit definition of the derivative to find $f'(x)$ for

$$f(x) = \frac{1}{2x + 3}.$$

- (6) (b) For the curve $y = x \ln x$, find the equation of the tangent line parallel to the line $2x - y = 5$.

(8) (c) Evaluate the following limits, if possible:

(i) $\lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$

(ii) $\lim_{x \rightarrow 0} \frac{3 - \sqrt{x + 9}}{x}$

(iii) $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x}$

(iv) $\lim_{x \rightarrow \infty} \frac{4x - 3 - 2x^2}{\sqrt{x} + x^2}$

(9) 4. Two balls rolling on a table strike one another. After the collision, the balls travel along perpendicular lines, one ball at 10 cm./sec., the other at 20 cm./sec.. How fast is the distance between the balls changing after 2 sec.?

(9) 5. A rectangle in the first quadrant has vertices A on the positive x -axis, then the origin O , then B on the positive y -axis, and finally C on the line $2x + 4y = 6$. Find the dimensions of the rectangle of largest possible area. Fully justify your answer using the methods of calculus.

6. Consider the graph of the function

$$f(x) = \frac{(x + 2)^2}{x^2 + 4}.$$

You are given that

$$f'(x) = \frac{16 - 4x^2}{(x^2 + 4)^2} \quad \text{and} \quad f''(x) = \frac{8x(x^2 - 12)}{(x^2 + 4)^3}.$$

(2) (a) Find the x -intercept.

(3) (b) Find all vertical or horizontal asymptotes, if any.

(3) (c) Determine the intervals where $f(x)$ is increasing or decreasing.

(3) (d) Determine the intervals where $f(x)$ is concave up or concave down.

(2) (e) Find all inflection points.

(3) (f) Find the absolute maximum value of $f(x)$, if it exists.

(5) (g) Sketch the graph of $f(x)$.

(4) 7. Do just one of the following questions. (Since only one will be marked, cross out any unwanted attempts.)

(a) Determine $\sin^{-1}(\sin(2))$. (Note: all angles in radians; 1 radian $\simeq 57.3^\circ$.)

(b) Determine all real numbers k such that

$$\lim_{x \rightarrow 0} \frac{x^k}{x^4} = 0 = \lim_{x \rightarrow \infty} \frac{x^k}{\sqrt{x^{11}}}.$$

(c) Given that $f' \left(\frac{1}{2} \right) = 2$, $f \left(\frac{1}{2} \right) = 3$ and $p(x) = \sin(x) \cdot f(\cos x)$, find $p' \left(\frac{\pi}{3} \right)$.