DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 1003

FINAL EXAMINATION — TIME: 3 HOURS
DECEMBER 2002 — TOTAL POINTS = 100

INSTRUCTIONS:

(a) You must show all calculations for full marks.

(b) Calculators **are not** permitted. (Huge calculations often mean you are on the wrong track!)

Part A

MARKS — **Answer all questions**

1. Find the derivative of each of the following functions **(do not simplify your answers):**

(4) (a) $f(x) = (1 + 4x^3)^{20}$

(4) (b) $y = x^3 + 3^x + \ln(3^x)$

(4) (c) $g(t) = \sinh(t) \sec(t)$

(4) (d) $y = x^{\sin x}$

(4) (e) $y = \ln(1 + 3x^2) + \sin^2(3x)$

2. Find the derivative of each of the following functions **(simplify your answers):**

(4) (a) $y = \dfrac{e^{2x} - 1}{x + 1}$

(4) (b) $f(x) = \sin^{-1}(\sqrt{1 - x^2})$

(8) 3. (a) Obtain the formula for $\dfrac{d}{dx} \cos^{-1} x$, given that

$$y = \cos^{-1} x \Longleftrightarrow x = \cos y \quad \text{and} \quad 0 \leq y \leq \pi.$$

You cannot simply quote the known formula! Your answer should be a function of x.

(b) Find an equation of the tangent line to to the curve with equation

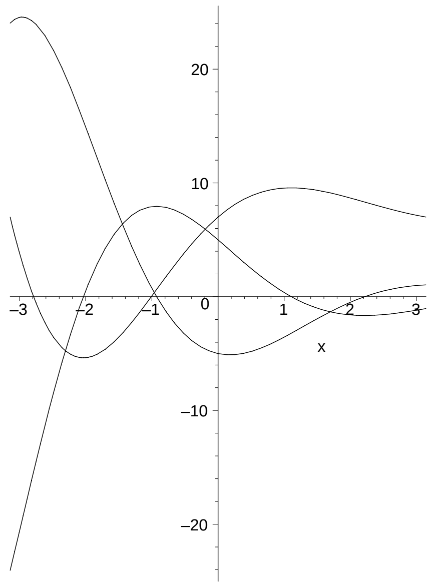
$$x^2 + xy + 2y^3 = 4,$$

at the point $(-2, 1)$.

(8) 4. Use the definition of the derivative as a limit to find the derivative of

$$f(x) = \sqrt{1 - x}.$$

(3) 5. The graphs of $f(x)$, $f'(x)$ and $f''(x)$ are shown. Label which graph is f, f' and f''.



(7) 6. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0} \frac{\tan 4x}{\sin 5x}$

(b) $\displaystyle\lim_{x\to\infty} \frac{x^2-4x-1}{x+2}$

(c) $\displaystyle\lim_{x\to 3} \left(\frac{1}{\sqrt{x+1}} - \frac{1}{2}\right)$

(8) 7. A water trough is 2 m deep, 3 m wide and 10 m long and it has a cross-section in the form of an inverted isosceles triangle. If water enters the tank at a rate of 3 m^3 per hour, how fast is the water level rising when the water is 1 m deep?

(8) 8. A cylindrical container with no top is to be constructed to hold 45 m^3 of liquid. The cost of the material used for the bottom is \$5 per m^2 and the cost of the material used for the curved face is \$3 per m^2. Use calculus to find the dimensions (radius and height) of the least expensive container. Justify why this is the least expensive.

9. Consider the function $f(x) = \dfrac{e^{-x}}{x-2}$. You are given that $f'(x) = \dfrac{(1-x)e^{-x}}{(x-2)^2}$ and $f''(x) = \dfrac{x^2-2x+2}{(x-2)^3}e^{-x}$.

(2) (a) Find the intercepts.

(3) (b) Find the vertical and horizontal asymptotes.

(3) (c) Determine where $f(x)$ is increasing or decreasing. Find all local maxima and minima.

(3) (d) Determine where $f(x)$ is concave up or concave down.

(3) (e) Use the above information to sketch the graph of $f(x)$.

Part B

Answer <u>four</u> of the following six questions for 4 marks each.

(4) 10. Does the graph of

$$y = \frac{\sin(x-1)}{x^2-1}$$

have a vertical asymptote at $x = 1$? Justify your answer.

(4) 11. Evaluate $\lim_{x \to 0} (1+3x)^{\frac{1}{x}}$.

(4) 12. Find the inverse of the function $f(x) = \dfrac{3x+2}{x+4}$. What is the domain and range of the inverse?

(4) 13. Find an equation of the tangent line to the graph of $y = \tan^{-1} 2x$ at the point where $x = \sqrt{3}/2$.

(4) 14. A particle moves in a straight line such that

$$s = 4\sin 3t, \quad t \geq 0,$$

where s is the displacement measured in meters and t is the time in seconds. What is the velocity of the particle at time t? What is the velocity at $t = 0$? When does the particle first come to rest? When does the particle first return to $s = 0$?

(4) 15. Suppose that $f(x)$ is a continuous and differentiable function such that $f(-1) = 2$ and $f'(x) \geq 4$ for $-1 \leq x \leq 2$. Use the mean value theorem to determine how small can $f(2)$ possibly be?