DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 1003

FINAL EXAMINATION
DECEMBER 2003

TIME: 3 HOURS
TOTAL POINTS = 100

INSTRUCTIONS:

(a) You must show all calculations for full marks.

(b) Calculators **are not** permitted.

MARKS

1. Find the derivatives of each of the following functions. (**Do NOT simplify answers!**)

(4) (a) $f(x) = \dfrac{8x}{\cos x + \sin x}$

(4) (b) $f(x) = \dfrac{\sqrt{9-x^2}}{x}$

(4) (c) $f(x) = e^{(x+1)} \ln(4x)$

(4) (d) $f(x) = \dfrac{\tan x + 5}{\sec x}$

(4) (e) $f(x) = \sin^{-1}(e^x)$ (same as $f(x) = \arcsin(e^x)$)

(4) (f) $f(x) = (\sinh x)\,(\cosh(3-x))$

(3) 2. Let $F(x) = f(g(x))$. If $f(2) = 3,\ f'(2) = 5,\ g(1) = 2$ and $g'(1) = 4$, find $F'(1)$.

3. Consider the curve defined by $x^2 + y^2 = x + y + 4$.

(4) (a) Find $\dfrac{dy}{dx}$ for this curve.

(1) (b) Verify that $(-1, 2)$ lies on the curve.

(3) (c) Find the equation of the tangent line to the curve at $(-1, 2)$.

(5) 4. Use logarithmic differentiation to find $\dfrac{dy}{dx}$ for $y = \dfrac{(4x+1)^6 \cdot \sqrt{x^3-3}}{x^9}$.

5. Evaluate the following limits:

(3) (a) $\lim\limits_{x\to\infty} \dfrac{x^2 - 4x^3}{1 + x^3}$

(3) (b) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 5x}$

(3) (c) $\lim\limits_{x\to 2^-} \dfrac{x^3 - 5}{x - 2}$

(3) (d) $\lim\limits_{x\to\infty} \dfrac{x^3}{e^x}$

(3) 6. Find a function f and a number a such that $f'(a) = \lim_{h \to 0} \dfrac{\sqrt[3]{8+h} - 2}{h}$.

7. Let $f(x) = \ln x - \ln(x-1)$.

(3) (a) Find the inverse function $f^{-1}(x)$.

(2) (b) Solve the equation $f(x) = 2$.

(4) 8. Using either a tangent line approximation or differentials, find a reasonable approximation to $\sqrt{63}$.

(5) 9. Sketch $y = x^2$. On your sketch, draw the two lines that pass through $(0, -1)$ and are tangent to $y = x^2$. Then, find (with justification) the coordinates of the two points of tangency.

10. Let $f(x) = \dfrac{x}{\sqrt{2 - x^2}}$.

(2) (a) What is the domain of f?

(2) (b) What are the vertical and horizontal asymptotes for f?

(1) (c) What is the range of f?

(2) (d) With a sketch, indicate the behaviour of $f(x)$ near the asymptotes.

11. Consider the function $f(x) = 4x^3 - x^4$.

(2) (a) Find all critical numbers of f.

(2) (b) Determine the intervals where $f(x)$ is increasing, where $f(x)$ is decreasing, and all local maxima and minima of $f(x)$, if any.

(2) (c) Find the intervals on which $f(x)$ is concave up, concave down, and the points of inflection, if any.

(4) (d) Sketch the graph of $y = f(x)$. Be sure that your graph indicates the information you have found in (a)–(c).

12. A heap of rubbish in the shape of a cube is being compacted into a smaller cube. The volume decreases at a rate of 2 cubic meters per minute.

(4) (a) Find the rate of change of an edge of the cube when the volume is exactly 27 cubic meters.

(4) (b) What is the rate of change of the surface area of the cube at that instant?

(6) 13. Consider the triangle in the xy-plane whose vertices are at $(0, 0)$, $(0, 1)$ and $(5, 0)$. Find the dimensions of the largest rectangle, with sides parallel to the x and y axis, that can be inscribed in the triangle. Justify your answer using the methods of calculus.