

# DEPARTMENT OF MATHEMATICS & STATISTICS

## MATH 1003

FINAL EXAMINATION  
DECEMBER 2003

TIME: 3 HOURS  
TOTAL POINTS = 100

### INSTRUCTIONS:

- (a) You must show all calculations for full marks.
- (b) Calculators **are not** permitted.

### MARKS

1. Find the derivatives of each of the following functions. (Do NOT simplify answers!)

(4) (a)  $f(x) = \frac{8x}{\cos x + \sin x}$

(4) (b)  $f(x) = \frac{\sqrt{9-x^2}}{x}$

(4) (c)  $f(x) = e^{(x+1)} \ln(4x)$

(4) (d)  $f(x) = \frac{\tan x + 5}{\sec x}$

(4) (e)  $f(x) = \sin^{-1}(e^x)$  (same as  $f(x) = \arcsin(e^x)$ )

(4) (f)  $f(x) = (\sinh x)(\cosh(3-x))$

(3) 2. Let  $F(x) = f(g(x))$ . If  $f(2) = 3$ ,  $f'(2) = 5$ ,  $g(1) = 2$  and  $g'(1) = 4$ , find  $F'(1)$ .

3. Consider the curve defined by  $x^2 + y^2 = x + y + 4$ .

(4) (a) Find  $\frac{dy}{dx}$  for this curve.

(1) (b) Verify that  $(-1, 2)$  lies on the curve.

(3) (c) Find the equation of the tangent line to the curve at  $(-1, 2)$ .

(5) 4. Use logarithmic differentiation to find  $\frac{dy}{dx}$  for  $y = \frac{(4x+1)^6 \cdot \sqrt{x^3-3}}{x^9}$ .

5. Evaluate the following limits:

(3) (a)  $\lim_{x \rightarrow \infty} \frac{x^2 - 4x^3}{1 + x^3}$

(3) (b)  $\lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 5x}$

(3) (c)  $\lim_{x \rightarrow 2^-} \frac{x^3 - 5}{x - 2}$

(3) (d)  $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$

- (3) 6. Find a function  $f$  and a number  $a$  such that  $f'(a) = \lim_{h \rightarrow 0} \frac{\sqrt[3]{8+h} - 2}{h}$ .
7. Let  $f(x) = \ln x - \ln(x - 1)$ .
- (3) (a) Find the inverse function  $f^{-1}(x)$ .
- (2) (b) Solve the equation  $f(x) = 2$ .
- (4) 8. Using either a tangent line approximation or differentials, find a reasonable approximation to  $\sqrt{63}$ .
- (5) 9. Sketch  $y = x^2$ . On your sketch, draw the two lines that pass through  $(0, -1)$  and are tangent to  $y = x^2$ . Then, find (with justification) the coordinates of the two points of tangency.
10. Let  $f(x) = \frac{x}{\sqrt{2-x^2}}$ .
- (2) (a) What is the domain of  $f$ ?
- (2) (b) What are the vertical and horizontal asymptotes for  $f$ ?
- (1) (c) What is the range of  $f$ ?
- (2) (d) With a sketch, indicate the behaviour of  $f(x)$  near the asymptotes.
11. Consider the function  $f(x) = 4x^3 - x^4$ .
- (2) (a) Find all critical numbers of  $f$ .
- (2) (b) Determine the intervals where  $f(x)$  is increasing, where  $f(x)$  is decreasing, and all local maxima and minima of  $f(x)$ , if any.
- (2) (c) Find the intervals on which  $f(x)$  is concave up, concave down, and the points of inflection, if any.
- (4) (d) Sketch the graph of  $y = f(x)$ . Be sure that your graph indicates the information you have found in (a)–(c).
12. A heap of rubbish in the shape of a cube is being compacted into a smaller cube. The volume decreases at a rate of 2 cubic meters per minute.
- (4) (a) Find the rate of change of an edge of the cube when the volume is exactly 27 cubic meters.
- (4) (b) What is the rate of change of the surface area of the cube at that instant?
- (6) 13. Consider the triangle in the  $xy$ -plane whose vertices are at  $(0, 0)$ ,  $(0, 1)$  and  $(5, 0)$ . Find the dimensions of the largest rectangle, with sides parallel to the  $x$  and  $y$  axis, that can be inscribed in the triangle. Justify your answer using the methods of calculus.