DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 2213

FINAL EXAMINATION
DECEMBER 1999 TIME: 3 Hours

ANSWER ALL QUESTIONS WITHOUT USE OF A CALCULATOR. CREDIT WILL BE GIVEN FOR PRESENTATION AND METHODS OF SOLUTIONS.

1. Use elementary row operations on matrices to find all solutions, if any, of the system of linear equations:

$$\begin{array}{rcrcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & + & x_4 & = & -5 \\ 4x_1 & + & 5x_2 & + & 2x_3 & - & x_4 & = & 4 \\ 4x_1 & + & 6x_2 & & & - & 5x_4 & = & 3 \\ 6x_1 & + & 7x_2 & + & x_3 & - & 4x_4 & = & 2. \end{array}$$

2. Given $A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & -2 & 7 \\ -1 & 2 & -1 \end{bmatrix}$, calculate $\det A$ and A^{-1}.

3. (i) Determine an LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 2 & -7 & -7 & -6 \\ -1 & 2 & 6 & 4 \\ -4 & -1 & 9 & 8 \end{bmatrix}.$$

   (ii) Use the decomposition of part (i) to rewrite the linear system

$$A\mathbf{x} = \mathbf{b}, \quad \mathbf{b} = [1, 7, 0, 3]^t$$

   as an equivalent pair of linear systems, one to be solved by back substitution and the other by forward substitution, but **do not** solve the sytem.

4. (i) Show that $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (3, 1, 1)$ and $\mathbf{u}_3 = (-6, 2, -2)$ form a basis in $\mathbb{R}^3$, and determine the coordinates of the vector $\mathbf{v} = (-4, 0, 0)$ with respect to the basis.

   (ii) Use the Gram-Schmidt process to determine the orthogonal basis in $\mathbb{R}^3$ that is given by the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of part (i).

5. (i) For the bases in $\mathbb{R}^3$,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}, \quad \mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \right\},$$

   determine the transition matrix, $P(\mathcal{C} \leftarrow \mathcal{B})$.

(ii) Given the following matrix A, and an echelon form of A,

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 \\ 3 & -7 & 8 & -5 & 8 \\ 3 & -9 & 12 & -9 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 \\ 0 & 2 & -4 & 4 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

find a basis for (a) the column space of A, and (b) the row space of A.

6. (i) A linear mapping $T : \mathbb{R}^4 \to \mathbb{R}^2$ is described as

$$T(\mathbf{x}) = (x_1 - x_2 + x_3 - x_4,\ \ 2x_1 - 2x_2 + 3x_3 - 3x_4)$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$. Deduce the matrix representation for T, and determine (as a span) the null space of T in $\mathbb{R}^4$.

(ii) For the matrix

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

find (a) all the eigenvalues of A, and (b) the eigenvectors of A corresponding to **the smallest eigenvalue**.