

DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 3033

FINAL EXAMINATION

DECEMBER 2001

TIME: 3 HOURS

MARKS

- (12) 1. Let $(Z, +)$ be the usual group of integers. Define a new binary operation on Z by $a * b = a + b + 2$. Show that $(Z, *)$ is a group.
- (14) 2. Consider the cyclic group $U(54)$.
- (a) What is its order?
 - (b) Show (efficiently!) that 5 is a generator.
 - (c) How many generators does $U(54)$ have?
 - (d) Find a generator for each subgroup of $U(54)$ and draw the subgroup lattice.
- (18) 3. In S_7 :
- (a) Write $\sigma = (23416)(5213)(7241)^{-1}$ as a product of disjoint cycles. What is $|\sigma|$?
 - (b) What are the possible orders for elements of S_7 ?
 - (c) What are the possible orders for elements of A_7 ?
 - (d) Let
$$A = \{\sigma \in S_7 \mid \sigma(2) = 2 \text{ and } \sigma(3) = 3\}$$
$$B = \{\alpha \in S_7 \mid \alpha(2) = 3 \text{ and } \alpha(3) = 2\}.$$
- (i) Prove that B is not a subgroup but both A and $A \cup B$ are subgroups of S_7 [where $A \cup B = \{\beta \mid \beta \in A \text{ or } \beta \in B\}$].
 - (ii) What is $|A|$? $|A \cup B|$?
- (11) 4. Determine the number of elements of each order in D_{21} and in $D_7 \oplus Z_3$. Could these groups be isomorphic?
- (11) 5. Consider $G = U(2160) = U(2^4 \cdot 3^3 \cdot 5)$.
- (a) Write G as a direct product of two groups in two different ways.
 - (b) Write G as a direct product of three groups.
 - (c) Using (b), determine the smallest n for which $x^n = 1$ for all $x \in G$.

- (12) 6. Define $\alpha : Z_{20} \rightarrow Z_{20}$ by $\alpha(x) = rx$ where $r \in Z_{20}$.
- (a) If $r = 3$ show α is an automorphism.
 - (b) If $r = 6$
 - (i) show α is a homomorphism but not an automorphism;
 - (ii) list the elements of $\ker \alpha$;
 - (iii) use the first isomorphism theorem to determine the order of $\alpha(Z_{20})$.
- (12) 7. Let H be a subgroup of G .
- (a) Prove $xH = H$ if and only if $x \in H$.
 - (b) Prove $xH = yH$ if and only if $x \in yH$.
 - (c) Prove for all x, y either $xH = yH$ or $xH \cap yH = \phi$.
- (10) 8. (a) One subgroup of $GL(n, \mathbb{R})$ is $H = \{A \mid \det A = 3^n \text{ for some } n \in \mathbb{Z}\}$. Prove $H \triangleleft GL(n, \mathbb{R})$.
- (b) Determine how many rotation symmetries each of the following solids has.

(100)

BONUS

9. Prove that $H < G$ is normal if and only if the following is true:

whenever $ab \in H$ then $ba \in H$.