DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 3033

FINAL EXAMINATION
DECEMBER 2002 TIME: 3 HOURS

MARKS

(12) 1. Let M be a fixed matrix in $G = GL(2, \mathbb{R})$. Define a binary operation $*$ on G by $A * B = AMB$. Show that $(G, *)$ is a group.

(10) 2. Draw the subgroup lattice for Z_{100} indicating the order of each subgroup and a generator for each.

(8) 3. If G is a group and $g \in G$ prove the centralizer of g $[C(g) = \{x | xg = gx\}]$ is a subgroup of G. Prove $\langle g \rangle \subseteq C(g)$.

(28) 4. In S_9:

(a) What is the largest order an element in S_9 can have?
(b) What is the largest order an element in A_9 can have?
(c) Write $\beta = (13)(17)(259)(289)$ as a product of disjoint cycles.
(d) Compute the elements of $H = \langle \beta \rangle$.
(e) Find $\text{stab}_H 1$, $\text{orb}_H 1$, $\text{stab}_H 2$, $\text{orb}_H 2$.
(f) Which of $\text{stab}_H 1$, $\text{stab}_H 2$ are normal in H and why?
(g) Find $\alpha \in C(\beta)$ with $\alpha \notin \langle \beta \rangle$.

(10) 5. Prove that $U(900)$ is isomorphic to $U(858)$ (given $858 = 6 \cdot 11 \cdot 13$).

(12) 6. How many elements are there of each order in D_{15} and $D_5 \oplus D_3$? Could these groups be isomorphic?

(10) 7. Let G be a group and $K = G \oplus G = \{(x, y) | x, y \in G\}$.
Let $H = \{(g, g) | g \in G\}$ (the "diagonal subgroup" of K).

(a) Prove that H is a subgroup of K.
(b) Prove that H is a normal subgroup of K if and only if G is abelian.

(10) 8. Recall from assignment 2 that the set of Heisenberg matrices

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$$

is a group under multiplication. Define $\varphi : G \to \mathbb{R} \oplus \mathbb{R}$ by

$$\varphi = \left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right) = (a, c)$$

where as usual $\mathbb{R} \oplus \mathbb{R}$ is a group under addition.

(a) Prove φ is a homomorphism.

(b) Prove φ is onto.

(c) Describe the elements of $\ker \varphi$.

BONUS:

(d) Prove $\ker \varphi = Z(G)$.