

DEPARTMENT OF MATHEMATICS & STATISTICS

MATH 3503

FINAL EXAMINATION

April, 2006

NO CALCULATORS

Time: 3 HOURS

MARKS

- (7) 1. (a) Use undetermined coefficients to find the general solution to the differential equation:

$$y'' + y = 3 \cos x + 4 \sin x.$$

- (7) (b) Use variation of parameters to solve

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 2e^{-3x}.$$

- (7) 2. A spring with a 4 kg mass attached to it has spring constant 36 N/m. There is no damping and an external force $F(t) = 12 \sin 3t$ N acts on the mass, where t is time. The mass is initially stretched 1 m beyond the equilibrium position and given an initial velocity of 2 m/s. Use the Laplace transform to find the position $x = x(t)$ of the mass measured from the equilibrium position.

3. Find the Laplace transform of the following functions.

(3) (a) $f(t) = (3t + 4)u_2(t)$.

(4) (b) $f(t) = \begin{cases} t^2, & \text{if } 0 < t < 2 \\ 0, & \text{if } t > 2 \end{cases}$

(4) 4. (a) Find the inverse Laplace transform of $F(s) = \frac{2s + 3}{(s^2 + 4s + 13)}e^{-3s}$.

- (3) (b) Use the convolution theorem to find

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 3)(s + 4)^3} \right\}. \quad \text{Do not evaluate the integral.}$$

- (3) 5. (a) Find $\mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} t & \text{if } 0 < t < 2 \\ 2 & \text{if } t > 2 \end{cases}$$

- (4) (b) Use the Laplace transform to solve

$$\ddot{x} + 4x = f(t), \quad x(0) = 1, \quad \dot{x}(0) = 0,$$

where $f(t)$ is the function of part (a).

- (7) 6. Find the general solution to the system of differential equations

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \text{where } A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 5 \end{bmatrix}$$

given that the characteristic equation for A is $-(\lambda - 2)^3 = 0$.

- (7) 7. Find the general solution to the system in terms of real functions

$$\frac{dx}{dt} = 3x - 2y$$

$$\frac{dy}{dt} = 4x - y$$

- (7) 8. Find the general solution to the non-homogeneous system of differential equations

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ e^t \end{bmatrix},$$

given that

$$\vec{x}^{(1)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \quad \text{and} \quad \vec{x}^{(2)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

are two linearly independent solutions to the corresponding homogeneous system.

- (7) 9. (a) Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & \text{if } -2 < x < 0 \\ 1, & \text{if } 0 < x < 2 \end{cases}$$

Sketch the graph of the function to which the series converges for $-6 \leq x \leq 6$.

- (7) (b) Find the Fourier cosine series for

$$f(t) = \frac{2A}{T}t, \quad \text{if } 0 < t < \frac{T}{2}$$

where T and A are positive constants. Sketch the graph of the function to which the series converges for $-T \leq t \leq 2T$.

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