Overview of research area: association schemes

Three closely related areas:

- Designs – experimental designs in statistics, especially $t$-designs
- Codes – information transmission
- Graphs – general modelling of networks

Results and methods of one area are applied to another.

Strongly regular graphs (srg’s)

- Symmetry inherent in $t$-designs is reflected in srg’s.
- Interaction between these is fruitful.
- Can be represented by matrices with nice algebraic properties.
An example would help

Example to illustrate the interplay between designs and srg’s.

- Start with a pentagon, an srg.
- Construct from this a $2 - (n, 3, \lambda)$ design.
- Construct the “block graph” of the design.
- The graph is a (different) srg.

Construct the design

- The blocks of the design are sets of 3.
- Choose these according to criterion: number of connections in pentagon graph is odd.
All possible sets of 3:

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Sets of 3 that satisfy the condition:

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These sets are the blocks of a $2 - (6, 3, 2)$ design.
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Construct the block graph

- Make the blocks into points of a new graph.
- Two are connected if they intersect in 2 points.
- This graph is strongly regular!
- Its sibling is known as the Triangular Graph, below.
Where do the association schemes come in?

- The two complementary block graphs above, along with the “equals” graph, form an association scheme.
- srg’s are the smallest rank association schemes.
- Studied in the 1950’s by Bose, Shimamoto, and others.
- Important in group theory in the 1970’s. Tie in to coding theory here.
- Delsarte in ’73 united the sphere-packing bound (major result in coding) and Fisher’s inequality (major result in designs) using association schemes.
- Result shows association schemes to be the basic underlying structure of both.
So what is an association scheme?

- Set of points, $X$ and some number of (symmetric) relations.
- Think of each relation as a graph.
- Three properties are essential:
  1. Every pair of points is joined in (exactly) one of the graphs.
  2. One of the relations is equality.

Third property requires some terminology.
Given points $x$ and $z$ that are joined in graph $k$, count triangles:

![Triangle Diagram]

Here, everything is fixed except $y$.

Call this number $p_{ij}^k(x, z)$. The third property is:

3. $p_{ij}^k(x, z)$ is independent of the choice of $x$ and $z$.

Call these numbers $p_{ij}^k$. They are the parameters of the association scheme.
Equivalent algebraic formulation

Let $A_i = $ matrix of the $i^{th}$ relation, or (equivalently) the adjacency matrix of the graph. These are square matrices, with rows and columns indexed by the points, and $(x, y)$ entry equalling 1 or 0, according as $x$ and $y$ are $i^{th}$ associates or not.

\[(i) \quad A_0 = I\]
\[(ii) \quad \sum A_i = J\]
\[(iii) \quad A_i \cdot A_j = \sum_k p_{ij}^k A_k\]

Importance of the matrix formulation:

- The linear span of the matrices is closed under multiplication.
- They form a basis of an associative algebra called the Bose-Mesner algebra.
Weighted association schemes

Weight the edges in each graph so that:

- Something new (interesting?) is obtained.
- Nice algebraic properties are retained.

Main result

- Put a weight as above on a known family of srg’s.
- Use values $\pm 1$.
- Determine when this is possible, and what the resulting structures are.
The following theorem shows that all non-trivial regular rank 3 weights on $L_2(n)$ are obtained from 2–designs.

**Theorem** If $\omega$ is a non-trivial regular weight with values $\pm 1$ and full support on the lattice graph $L_2(n)$ then $n$ is even and $\omega = \omega_1 \otimes \omega_2$, where $\delta \omega_1$ and $\delta \omega_2$ are regular 2–graphs with the same parameters.

Note: A regular 2–graph is a $2 - (n, 3, \lambda)$ design.
Example of a regular weight

Let $\Gamma$ be the lattice graph $L_2(6)$. Let $A_1$, $A_2$ be adjacency matrices of $\Gamma$ and $\overline{\Gamma}$ respectively. Put

$$C = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 & 1 & -1 \\ 1 & 1 & -1 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & 0 \end{pmatrix}. $$

$C$ is a conference matrix of order 6 ($C^2 = 5I$), hence represents a regular 2–graph on 6 vertices. Put

$$\omega = (I + C) \otimes (I + C)$$

$$= I \otimes I + I \otimes C + C \otimes I + C \otimes C.$$