Some suitable questions from old exams
in Math 2213, Math 2503, Math 2513

(see http://www.math.unb.ca/exams)

Find the general solution for

$$
\begin{array}{rcrcrcrcr}
-x_1 & + & 2x_2 & + & x_3 & - & x_4 & = & 1 \\
x_1 & - & 4x_2 & & & + & 5x_4 & = & -2 \\
-2x_1 & + & 6x_2 & - & x_3 & - & 5x_4 & = & -4 \\
-x_1 & - & 4x_2 & + & 4x_3 & + & 11x_4 & = & -2
\end{array}.
$$

(10) 2. (i) Compute the inverse of the matrix

$$
A = \begin{bmatrix} 2 & -1 & -4 \\ -1 & 1 & 2 \\ -1 & 1 & 3 \end{bmatrix}.
$$

(ii) Use the result of part (i) to find the inverses of the three matrices A^t, $3A$ and A^2.

(10) 3. Let $A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 & 3 \\ 2 & 4 & -3 & 2 & 0 \\ -3 & -6 & 2 & 0 & 3 \end{bmatrix}$.

Ignore in Math 1503

(i) Determine a basis for the column space of A.

(ii) ~~If $T : \mathbb{R}^5 \to \mathbb{R}^4$ is the linear transformation induced by A, d~~Describe the nullspace of T in terms of a span in $\mathbb{R}^5$.

(10) 4. (a) Let S denote the subset of all vectors $\vec{x}$ in $\mathbb{R}^2$ for which $\vec{x} \cdot \vec{u} = 0$ for some fixed $\vec{u} \in \mathbb{R}^2$. Is S a subspace of $\mathbb{R}^2$ or not? Give reasons for your answer.

Ignore for Math 1503

(10) 5. (a) Find the eigenvalues and corresponding eigenvectors for

$$
A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.
$$

—OVER—

(3) 4. If $A = \begin{bmatrix} 1 & -1 & 6 \\ 1 & -2 & -1 \\ 3 & -3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 \\ 2 & -4 \\ -3 & 1 \end{bmatrix}$, find AB and BA, where possible.

(6) 5. Use Gauss-Jordan elimination to solve the system

$$\begin{array}{rcrcrcrcrcr} 2x_1 & - & 4x_2 & + & 10x_3 & + & 3x_4 & - & 8x_5 & = & 4 \\ x_1 & + & 2x_2 & + & x_3 & - & 2x_4 & & & = & 7 \\ 3x_1 & + & 5x_2 & + & 4x_3 & - & 10x_4 & - & x_5 & = & 10 \end{array}.$$

(6) 6. Find the inverse of the matrix

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix}.$$

(3) 7. If $A = \begin{bmatrix} t^3 & 0 & t^2 \\ 17 & 0 & 1 \\ t^4 & 2 & -17 \end{bmatrix}$, for what values of t does A^{-1} not exist?

(10) 8. Determine the eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 1 & 8 & -2 \\ 1 & 0 & 0 \\ 0 & 4 & -1 \end{bmatrix}.$$

1. Given $\vec{a} = (1, -1, 1)$, $\vec{b} = (2, 3, 0)$ and $\vec{c} = (5, 0, 12)$ calculate the following:

(3) (a) the cosine of the angle between $\vec{b}$ and $\vec{c}$;

(3) (b) the vector projection of $\vec{a}$ onto $\vec{c}$;

(3) (c) $\vec{c} \cdot (\vec{a} \times \vec{b})$

(4) 2. (a) Find the parametric **and** symmetric equations of the line which is parallel to the line of intersection of the planes $x + y = 2$ and $-2x + z = 0$, and which passes through the origin.

(3) (b) Find the equation of the plane which contains the point $(1, 0, 0)$ and the line $L: \dfrac{x-1}{2} = y + 1 = \dfrac{z}{3}$.

Answers to Practice Problems (maybe not error free; but you can get the idea)

(1). Reduce aug. matrix

$$\left[\begin{array}{cccc|c} -1 & 2 & 1 & -4 & 1 \\ 1 & -4 & 0 & 5 & -2 \\ -2 & 6 & -1 & -5 & -4 \\ -1 & -4 & 4 & 11 & -2 \end{array}\right]$$

to

$$\begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -4 & 7 \\ 0 & 1 & 0 & -9/4 & 9/4 \\ 0 & 0 & 1 & -1/2 & 7/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4\ (\text{Free}) \end{array} \end{array}$$

General sol^n

$$\vec{x} = \begin{bmatrix} 7 \\ 9/4 \\ 7/2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 9/4 \\ 1/2 \\ 1 \end{bmatrix}$$

(2) Reduce $\left[\begin{array}{ccc|ccc} 2 & -1 & -4 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ -1 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$ so $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

to $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$

$$(A^T)^{-1} = (A^{-1})^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$(3A)^{-1} = \frac{1}{3} A^{-1} = \begin{bmatrix} 1/3 & -1/3 & \frac{2}{3} \\ 1/3 & \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix}$$

$$(A^2)^{-1} = (A^{-1})^2 = \begin{bmatrix} 0 & -5 & 4 \\ 3 & 3 & 2 \\ -1 & -3 & 1 \end{bmatrix}$$

(3) Reduce

$$A = \begin{array}{c} \begin{array}{ccccc} \checkmark & & \checkmark & \checkmark & \end{array} \\ \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ -1 & -2 & 1 & 2 & 3 \\ 2 & 4 & -3 & 2 & 0 \\ -3 & -6 & 2 & 0 & 3 \end{bmatrix} \end{array}$$

to

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(i) IMP - use pivot columns in orig. matrix A to get basis for col. space

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

(ii) nullspace is spanned by the two special solutions

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(4) S is a subspace:

$\vec{a}, \vec{b}$ in S means $\vec{a}\cdot\vec{u} = 0$, $\vec{b}\cdot\vec{u} = 0$, so $(\vec{a}+\vec{b})\cdot\vec{u} = 0+0 = 0$

$(t\vec{a})\cdot\vec{u} = 0$

Geometrically, S is just a line through $\vec{0}$ when $\vec{u} \neq \vec{0}$, or $S = \mathbb{R}^2$ when $\vec{u} = \vec{0}$.

(5)

$$0 = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)[\lambda^2 - 2\lambda] - [2-\lambda] + [2-\lambda]$$

$$= (1-\lambda)(\lambda)(\lambda-2)$$

So eigenvalues

$\lambda = 0$ $\quad$ $\lambda = 1$ $\quad$ $\lambda = 2$

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

Reduce to

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvectors

$$\vec{x} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{x} = x_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{x} = x_3\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(6) Can only find

$$AB = \begin{bmatrix} -15 & 13 \\ 4 & 10 \\ 21 & 17 \end{bmatrix}$$

(5) Reduce any matrix to

$$\begin{array}{ccccc|c} & & \text{Free} & & \text{Free} & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & -2 & 5 \\ 0 & 1 & -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{array}\right]$$

Gen. solution $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(6) Inverse = $\begin{bmatrix} 1 & -1 & 1 \\ -6 & 8 & -5 \\ -4 & 5 & -3 \end{bmatrix}$

(7)

expand down

$$\det A = \begin{vmatrix} t^3 & 0 & t^2 \\ 17 & 0 & 1 \\ t^4 & 2 & -17 \end{vmatrix} = (-0)(\ ) + (0)(\ ) - 2\left[t^3 - 17t^2\right]$$

$$= -2t^2(t-17)$$

We ~~do~~ know A^{-1} d.n.e. when this $= 0$.

Answer: $\underline{t = 0 \text{ or } 17}$

(8)

$$0 = \begin{vmatrix} (1-\lambda) & 8 & -2 \\ 1 & -\lambda & 0 \\ 0 & 4 & (-1-\lambda) \end{vmatrix} = -1\left[-8-8\lambda+8\right] - \lambda\left[(1-\lambda)(-1-\lambda)\right]$$

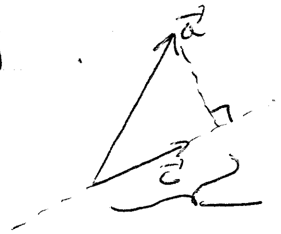
$$= 8\lambda - \lambda\left[\lambda^2 - 1\right] = \lambda\left[9 - \lambda^2\right], \text{ so } \lambda = 0, 3, -3$$

eigenvectors:

$\lambda = 0$	$\lambda = 3$	$\lambda = -3$
$\vec{x} = x_3 \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$	$\vec{x} = x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$	$\vec{x} = x_3 \begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix}$

(1) (a) $\cos\theta = \dfrac{\vec{b}\cdot\vec{c}}{\|\vec{b}\|\,\|\vec{c}\|} = \dfrac{10}{\sqrt{13}\,\sqrt{169}} = \dfrac{10}{13\sqrt{13}}$

(b)



vector of projection $= \left(\dfrac{\vec{a}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}\right)\vec{c} = \dfrac{17}{169}\,(5,0,12)$

(c)

$$\vec{a} = (1,-1,1)$$
$$\vec{b} = (2,3,0)$$
$$(\vec{a}\times\vec{b}) = (-3,2,5)$$

so $\vec{c}\cdot(\vec{a}\times\vec{b}) = (-3)(5) + 12(5)$

$= \underline{\underline{45}}$

(2) (a) Line of intersection of planes: solve $\begin{aligned} x+y &= 2 \\ -2x + z &= 0 \end{aligned}$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ -2 & 0 & 1 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & 2 \end{array}\right]$

so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + z \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

poss. direction vector $= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$; alternatively, take cross prod of plane normals $(1,1,0)\times(-2,0,1) = (1,-1,2)$

Para. Equ'ns for parallel line through $\vec{0}$: $\begin{cases} x = \frac{1}{2}t \\ y = -\frac{1}{2}t \\ z = t \end{cases}$ $\longleftrightarrow$ Symm. Equ'ns: solve for t

$\dfrac{x-0}{\frac{1}{2}} = \dfrac{y-0}{-\frac{1}{2}} = \dfrac{z-0}{1}$

(we didn't really do this!)

(b) Let each $\dfrac{x-1}{2} = y+1 = \dfrac{z}{3} = t$, so $\begin{cases} x = 1+2t \\ y = -1+t \\ z = 0+3t \end{cases}$

Take $t = 0, 1$: so our plane must contain $(1,-1,0)$, $(3,0,3)$, as well as $(1,0,0)$

We get $\underline{3x - 2z = 3}$