Regions of absolute stability

The purpose of this worksheet is to plot regions of absolute stability for several of the one-step stencils we have previously considered. In particular, we are looking at the forward Euler, backward Euler, trapezoidal, Huen and classical 4th order Runge-Kutta (as usual the ODE we consider is $y' = f(x,y)$):

```
> restart;

with(plots):
```

| animate, animate3d, animatecurve, arrow, changecoords, complexplot, complexplot3d, |
| (1) |
| conformal, conformal3d, contourplot, contourplot3d, coordplot, coordplot3d, densityplot, |
| display, dualaxisplot, fieldplot, fieldplot3d, gradplot, gradplot3d, implicitplot, implicitplot3d, |
| inequal, interactive, interactiveparams, intersectplot, listcontplot, listcontplot3d, |
| listdensityplot, listplot, listplot3d, loglogplot, logplot, matrixplot, multiple, odeplot, pareto, |
| plotcompare, pointplot, pointplot3d, polarplot, polygonplot, polygonplot3d, |
| polyhedra_supported, polyhedraplot, rootlocus, semilogplot, setcolors, setoptions, |
| setoptions3d, spacecurve, semiparametricplot, surfdata, textplot, textplot3d, tubeplot |

```
> Forward := y_new = y_old+f(x_old,y_old)*h;
Backward := y_old-y_new+f(x_new,y_new)*h = 0;
Trap := y_new-y_old-1/2*f(x_old,y_old)*h-1/2*f(x_new,y_new)*h = 0;
Huen := y_new = y_old+h*(1/2*f(x,y_old)+1/2*f(x+h,y_old+f(x,y_old)));
RK4 := y_new = y_old+h*(1/6*f(x,y_old)+1/3*f(x+1/2*h,y_old+1/2*h*f(x,y_old)) + 1/3*f(x+1/2*h,y_old+1/2*h*f(x,y_old)) + 1/2*h*f(x+h, y_old+f(x,y_old) + 1/2*h*f(x+h, y_old+1/2*h*f(x+h, y_old+1/2*h*f(x+h, y_old+1/2*h*f(x+h, y_old))));

Forward := y_new = y_old + f(x_old, y_old) h  
Backward := y_old - y_new + f(x_new, y_new) h = 0  
Trap := y_new - y_old - 1/2 f(x_old, y_old) h - 1/2 f(x_new, y_new) h = 0  
Huen := y_new = y_old + h \left(1/2 f(x, y_old) + 1/2 f(x + h, y_old + f(x, y_old) h) \right) 
RK4 := y_new = y_old + h \left(1/6 f(x, y_old) + 1/3 f(x + 1/2 h, y_old + 1/2 f(x, y_old) h) + 1/3 f(x + 1/2 h, y_old + 1/2 h f(x, y_old)) + 1/2 h f(x + 1/2 h, y_old + 1/2 h f(x + 1/2 h, y_old + 1/2 h f(x + 1/2 h, y_old + 1/2 h f(x + 1/2 h, y_old)))) \right) 
```

We are going to consider the test problem defined by:

```
> f := (x, y) -> lambda*y;
```

```
f := (x, y) -> \lambda y
```

In this case, we can solve each stencil for $y_{\text{new}}/y_{\text{old}}$ in terms of $\lambda \cdot h$; i.e., $y_{\text{new}} = R(\lambda \cdot h)$.
*y_old:

\[
\text{Forward} := \frac{\text{y_new}}{\text{y_old}} = 1 + \lambda h \hfill (4)
\]

\[
\text{Backward} := \frac{\text{y_new}}{\text{y_old}} = -\frac{1}{-1 + \lambda h} \hfill (4)
\]

\[
\text{Trap} := \frac{\text{y_new}}{\text{y_old}} = -\frac{2 + \lambda h}{-2 + \lambda h} \hfill (4)
\]

\[
\text{Huen} := \frac{\text{y_new}}{\text{y_old}} = 1 + \lambda h + \frac{1}{2} \lambda^2 h^2 \hfill (4)
\]

\[
\text{RK4} := \frac{\text{y_new}}{\text{y_old}} = 1 + \lambda h + \frac{1}{2} \lambda^2 h^2 + \frac{1}{6} \lambda^3 h^3 + \frac{1}{24} \lambda^4 h^4 \hfill (4)
\]

It is useful to organize these stencils in a list:

\[
\text{stencils} := \text{[Forward, Backward, Trap, Huen, RK4];}
\]

\[
\text{stencil_names} := \text{["Forward Euler", "Backward Euler", "Trapezoidal", "Huen method", "Classical 4 order Runge-Kutta"];}
\]

\[
\text{stencils} := \left[\frac{\text{y_new}}{\text{y_old}} = 1 + \lambda h, \frac{\text{y_new}}{\text{y_old}} = -\frac{1}{-1 + \lambda h}, \frac{\text{y_new}}{\text{y_old}} = -\frac{2 + \lambda h}{-2 + \lambda h}, \frac{\text{y_new}}{\text{y_old}} = 1 \right.
\]

\[
+ \lambda h + \frac{1}{2} \lambda^2 h^2, \frac{\text{y_new}}{\text{y_old}} = 1 + \lambda h + \frac{1}{2} \lambda^2 h^2 + \frac{1}{6} \lambda^3 h^3 + \frac{1}{24} \lambda^4 h^4 \left.\right]\]

\[
\text{stencil_names} := \text{["Forward Euler", "Backward Euler", "Trapezoidal", "Huen method", "Classical 4 order Runge-Kutta"]}
\]

The region of absolute stability for each of the stencils is defined as the region in the complex \( z = h^* \) lambda plane for which the righthand sides have magnitudes less than one. These regions are plotted below in red.

\[
\text{for i from 1 to 5 do:}
\text{contourplot(simplify(subs(lambda=z/h,z=x+I*y,abs(rhs(stencils[i])))),x=-3.5..3.5,y=-3.5..3.5,contours=[1],grid=[50,50],axes=boxed,filled=true,title=stencil_names[i],labels=["Re(z)","Im(z)"])};
\text{od;}
\]
Huen method
Classical 4 order Runge-Kutta